COMPACT OSSERMAN MANIFOLDS WITH NEUTRAL METRIC

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Abstract. It is shown that if a compact four-dimensional manifold with metric of neutral signature is Jordan-Osserman, then it is either of constant sectional curvature or Ricci flat.

1. Introduction

Self-dual Einstein metrics are of special interest both in Physics and Geometry (see for example [2, 9, 10, 16] and the references therein). From the point of view of the curvature, self-dual Einstein metrics are characterized by the pointwise Osserman property (the eigenvalues of the Jacobi operators are constant on the unit pseudo-spheres at each point $p \in M$). Four-dimensional Osserman metrics of Riemannian signature are locally real or complex space forms, while Lorentzian Osserman metrics are necessarily of constant curvature (see [18, 19] for more information). On the other hand, there are plenty of examples of Osserman metrics in signature $(--++)$ (cf. [3, 8, 12, 15]). The Jordan normal form plays a crucial role in the higher signature setting – a self-adjoint linear transformation need not be determined by its eigenvalues if the metric in question is indefinite. One says that M is *Jordan-Osserman* if the Jordan normal form of the Jacobi operators is constant on the pseudo-sphere bundles $S^{\pm}(TM)$. Although the spacelike and timelike Jordan-Osserman conditions are equivalent in signature $(--++)$ they do not necessarily imply the null Jordan-Osserman condition (see [17] for details).

The relationship between local and global geometric properties is well developed in Riemannian geometry. In contrast, in areas such as Lorentzian or more generally pseudo-Riemannian geometry, little is known about global properties of the geometry. In most results, the sign of the curvature (sectional curvature, Ricci curvature, etc.) plays a fundamental role; the celebrated Calabi-Markus Theorem [7] is one such example. However, such assumptions have no sense in the case of pseudo-Riemannian metrics of neutral signature, since one may move from positive to negative curvature by reversing the sign of the original metric. The simplest case to be considered is that of four-dimensional $(--++)$-metrics, where some generalizations of the Hitchin-Thorpe inequality for Einstein metrics have been developed by Law and Matsushita [24, 25].

The following is one of the two main results of this paper. It shows that compactness is a strong restriction when dealing with Osserman metrics.

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Theorem 1. *Let (M,g) be a compact simply connected Jordan-Osserman manifold with metric of signature $(--++)$. Then it is of constant sectional curvature or otherwise the Jacobi operators are two-step nilpotent.*

Further observe that the Jordan normal form of the Jacobi operators may change from point to point in an Osserman manifold. Hence, we have

Theorem 2. *Let (M,g) be a compact simply connected Osserman manifold with metric of signature $(--++)$. Then $\tau[M]=0$ and the Jacobi operators have only one eigenvalue, which may be a single, double or triple root of the minimal polynomial. Moreover $\chi[M]\leq 0$, and $\chi[M]=0$ if and only if the Jacobi operators are nilpotent.*

Through this paper we will construct some additional structures (almost para-Hermitian and almost anti-Hermitian ones) on Osserman manifolds by means of the conformal Weyl curvature operator. We shall establish Theorems 1 and 2 by analyzing topological obstructions to the existence of such structures.

2. Osserman metrics and self-dual structures

For any non-null vector X, the Jacobi operator $\mathcal{J}(X)=R(X,\cdot)X$ can be viewed as a self-adjoint operator on the Lorentzian vector space $\langle X\rangle^\perp$. The following possibilities may occur at the algebraic level [3]:

Type Ia: the Jacobi operators are diagonalizable in an orthonormal basis,
Type Ib: the Jacobi operators have a complex eigenvalue,
Type II: the Jacobi operators are a double root of their minimal polynomial,
Type III: the Jacobi operators are a triple root of their minimal polynomial.

Considering the curvature tensor R as an endomorphism of $\Lambda^2(M)$, we have the $O(2,2)$-decomposition $R\equiv\frac{Sc}{12}\,\mathrm{Id}_{\Lambda^2}+Ric_0+W:\ \Lambda^2\to\Lambda^2$, where Ric_0 denotes the traceless Ricci tensor, $Ric_0(X,Y)=Ric(X,Y)-\frac{Sc}{4}\,g(X,Y)$. The Hodge star operator $\star:\Lambda^2\to\Lambda^2$ associated to any $(--++)$-metric induces a further splitting $\Lambda^2=\Lambda^2_+\oplus\Lambda^2_-$, where $\Lambda^2_\pm$ denotes the ±1-eigenspaces of the Hodge star operator, that is, $\Lambda^2_\pm=\{\alpha\in\Lambda^2(M)/\star\alpha=\pm\alpha\}$. Correspondingly, the curvature tensor further decomposes as

$$R\equiv\frac{Sc}{12}\,\mathrm{Id}_{\Lambda^2}+Ric_0+W^++W^-, \tag{1}$$

where $W^\pm=\frac{1}{2}\left(W\pm\star W\right)$. Recall that a pseudo-Riemannian 4-manifold is called *self-dual* (resp., *anti-self-dual*) if $W^-=0$ (resp., $W^+=0$).

The connection between Osserman and (anti-) self-dual metrics relies on the following Fact, and thus the analysis of the (anti-) self-duality conditions will play a basic role in what follows.

Fact 3. [1, 18] *A four-dimensional pseudo-Riemannian manifold is pointwise Osserman if and only if it is Einstein self-dual (or anti-self-dual).*

Moreover, there is a one to one correspondence among the Jordan normal forms of the Jacobi operators and the corresponding ones for $W^\pm$ as follows. Take a local orthonormal basis of vector fields $\{e_1,e_2,e_3,e_4\}$ on M. Then, local bases of the spaces of self-dual and anti-self-dual two-forms are as follows

$$E_1^\pm=\frac{e^1\wedge e^2\pm e^3\wedge e^4}{\sqrt{2}},\quad E_2^\pm=\frac{e^1\wedge e^3\pm e^2\wedge e^4}{\sqrt{2}},\quad E_3^\pm=\frac{e^1\wedge e^4\mp e^2\wedge e^3}{\sqrt{2}}.$$

Further note that $\langle E_1^\pm, E_1^\pm\rangle = 1$, $\langle E_2^\pm, E_2^\pm\rangle = -1$, $\langle E_3^\pm, E_3^\pm\rangle = -1$, and therefore the self-dual and anti-self-dual Weyl curvature operators $W^\pm : \Lambda^2_\pm \longrightarrow \Lambda^2_\pm$ may have the same Jordan canonical forms, as previously considered for the Jacobi operators. Furthermore, assuming $W^- = 0$ (a completely analogous analysis will give the anti-self-dual case) the following algebraic correspondence, with respect to the basis above, is obtained after a straightforward calculation.

Type Ia: Diagonalizable Jacobi operators with eigenvalues α, β, γ correspond to diagonalizable self-dual Weyl curvature operator

$$W^+ = \begin{pmatrix} 2\alpha - \frac{Sc}{6} & 0 & 0 \\ 0 & 2\beta - \frac{Sc}{6} & 0 \\ 0 & 0 & -2(\alpha+\beta) + \frac{Sc}{3} \end{pmatrix}.$$

Type Ib: Jacobi operators have eigenvalues α, $\gamma \pm \beta\sqrt{-1}$ if and only if the self-dual Weyl curvature operator satisfies

$$W^+ = \begin{pmatrix} \frac{2}{3}(\gamma-\alpha) & -2\beta & 0 \\ 2\beta & \frac{2}{3}(\gamma-\alpha) & 0 \\ 0 & 0 & \frac{4}{3}(\alpha-\gamma) \end{pmatrix}.$$

Type II: Jacobi operators have two eigenvalues α and β, the later being a double root of the minimal polynomial if and only if the self-dual Weyl curvature operator satisfies

$$W^+ = \begin{pmatrix} \frac{2}{3}(\beta-\alpha)+1 & -1 & 0 \\ 1 & \frac{2}{3}(\beta-\alpha)-1 & 0 \\ 0 & 0 & \frac{4}{3}(\alpha-\beta) \end{pmatrix}.$$

Type III: Jacobi operators have exactly one eigenvalue α, which is a triple root of the minimal polynomial if and only if the self-dual Weyl curvature operator satisfies

$$W^+ = \begin{pmatrix} 0 & 0 & \sqrt{2} \\ 0 & 0 & \sqrt{2} \\ -\sqrt{2} & \sqrt{2} & 0 \end{pmatrix}.$$

Remark 4. Recall that a four-dimensional metric is Einstein if and only if the decomposition (1) becomes $R \equiv \frac{Sc}{12}\,\mathrm{Id}_{\Lambda^2} + W^+ + W^-$. In such a case, the Euler characteristic $\chi[M]$ and the Hirzebruch signature $\tau[M]$ are expressed as follows [24]

$$\begin{aligned} \chi[M] &= -\frac{1}{8\pi^2}\int_M \left\{ \mathrm{tr}[(W^+)^2] + \mathrm{tr}[(W^-)^2] + \frac{Sc^2}{24} \right\} v, \\ \tau[M] &= \frac{2}{3}\frac{1}{8\pi^2}\int_M \left\{ \mathrm{tr}[(W^+)^2] - \mathrm{tr}[(W^-)^2] \right\} v. \end{aligned} \tag{2}$$

Further observe from (2) that $\chi[M] \le 0$ for any compact Einstein $(--++)$–metric, provided that $W^\pm$ are not of Type Ib.

3. Proof of the Theorems

It is a fundamental fact that an orientable 4-manifold with a field of 2-planes (equivalently, a two-dimensional distribution) admits two kinds of almost complex structures that induce opposite orientations. Since a neutral 4-manifold, i.e., an indefinite 4-manifold of metric signature $(--++)$, admits a field of 2-planes, it is necessarily an almost Hermitian 4-manifold.

Fact 5. [26] *An orientable four-dimensional manifold admits a* $(--++)$*–metric if and only if it satisfies a pair of Wu's conditions as follows*

$$\begin{aligned} c_1^2[M] &= 3\tau[M] + 2\chi[M], \\ c_1^2[-M] &= 3\tau[-M] + 2\chi[-M] = -3\tau[M] + 2\chi[M], \end{aligned}$$

where $-M$ *stands for* M *with the opposite orientation.*

Hence, it follows from (2) the first Chern number $c_1^2[M]$ and the first opposite Chern number $c_1^2[-M]$ of an Einstein $(--++)$-manifold are given by

$$\begin{aligned} c_1^2[M] &= -\frac{1}{2\pi^2}\int_M \left\{ \operatorname{tr}[(W^-)^2] + \frac{Sc^2}{48} \right\} v \\ c_1^2[-M] &= -\frac{1}{2\pi^2}\int_M \left\{ \operatorname{tr}[(W^+)^2] + \frac{Sc^2}{48} \right\} v. \end{aligned} \tag{3}$$

Remark 6. The fundamental form of an indefinite almost Hermitian structure defines a smooth section of Λ^2_- of constant norm 2 and conversely, any smooth section Ω of Λ^2_- of constant norm 2 is the fundamental form of an indefinite almost Hermitian structure. The inner product on $\Lambda^2_\pm$ induced by a $(--++)$-metric has signature $(--+)$. This shows that there are two almost complex structures which induce opposite orientations; Fact 5 now follows.

3.1. **Proof of the Theorem 1.** Recall from Fact 3 that an algebraic curvature tensor is Osserman if and only if it is Einstein and self-dual or anti-self-dual. We assume in what follows that $W^- = 0$ (a completely analogous analysis will prove the anti-self-dual case). Hence, the self-dual part of the Weyl conformal curvature tensor corresponds to one of the Jordan canonical forms Ia–III.

Next, recall that (g, P) is said to be an almost paraHermitian structure if $P^2 = \mathrm{Id}$ and $g(PX, PY) = -g(X, Y)$ for all vector fields X, Y on M. Then the fundamental form $\Omega_P(X, Y) = g(PX, Y)$ defines a section of Λ^2_+ of constant norm -2. Conversely, any smooth section Ω of Λ^2_+ of constant norm -2 is the fundamental form of an almost paraHermitian structure (see also [22], where the inner product on Λ^2 is taken with the opposite sign that we have chosen).

Further, observe that the existence of an almost paraHermitian structure (g, P) is an equivalent condition to the existence of an almost anti-Hermitian structure (g, J) (i.e., $J^2 = -\mathrm{Id}$, $g(JX, JY) = -g(X, Y)$ for all vector fields X, Y on M). Indeed, let (g, P) be an almost paraHermitian structure and let h be a Riemannian metric on M such that $h(PX, PY) = h(X, Y)$ for all X, Y and $h(QX, Y) = g(X, Y)$ where Q is an almost product structure (i.e., $Q^2 = \mathrm{Id}$) and put $J = PQ$. An straightforward calculation shows that J is an almost complex structure on M and moreover

$$g(JX, JY) = g(PQX, PQY) = -g(QX, QY) = -g(X, Y),$$

which shows that (g, J) is almost anti-Hermitian. Moreover a straightforward calculation shows that J is h-orthogonal and that the fundamental forms Ω_P and $\Omega^h_J(X, Y) = h(JX, Y)$ coincide with each other. Hence, both P and J induce the same orientation on M.

Chern classes of almost complex manifolds with anti-Hermitian metrics were studied in [4, 5], showing that the existence of such structures is a much more restrictive condition than that of almost Hermitian ones, as shown in the following

Fact 7. [4] *Let* (M, g, J) *be an almost anti-Hermitian manifold. Then all odd Chern classes vanish (i.e.,* $c_{2k+1}[M] = 0$*, for all* k*).*

Next we consider the different possibilities for the self-dual Weyl conformal operators. First of all, assume W^+ to be of Type Ia. A complete solution for the Osserman problem is known in this case: either it is a space of constant sectional curvature, or it is an indefinite Kähler manifold of constant holomorphic sectional curvature or a paraKähler manifold of constant paraholomorphic sectional curvature [3]. Observe that in the last two cases there are exactly two-distinct eigenvalues of the Jacobi operators in a ratio $1 : \frac{1}{4}$. If M is a paracomplex space form, then it admits an almost anti-Hermitian structure and hence $c_1[M] = 0$. Then (3) shows that $Sc = 0$ since $W^- = 0$ and hence M is flat in the compact case. The situation for the indefinite complex space forms is somehow different, since they are anti-self-dual (instead of self-dual as it occurs in the positive definite case) and the distinguished eigenvalue of W^- has spacelike associated eigenspace. Moreover,

$$W^- = \frac{Sc}{12}\begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

and thus

$$\chi[M] = -\frac{1}{8\pi^2}\int_M \left\{ \mathrm{tr}[(W^+)^2] + \mathrm{tr}[(W^-)^2] + \frac{Sc^2}{24} \right\} v = -\frac{1}{8\pi^2}\int_M \frac{Sc^2}{12} v$$

which shows that $\chi[M] \leq 0$, with equality if and only if M is flat. Kähler-Einstein metrics have been investigated by Petean [27], showing that the possible non Ricci-flat ones occur as minimal ruled surfaces over curves of genus $\mathfrak{g} \geq 2$, or they occur as minimal surfaces of class VII_0. Now, since any VII_0-surface has nonnegative Euler characteristic, any such surface supports a Kähler Osserman metric if and only if $\chi[M] = 0$, and the metric is flat. Next, assume M to be a minimal ruled surface over a curve of genus $\mathfrak{g} \geq 2$. Then the Chern numbers $c_1^2[M]$ and $c_2[M]$ satisfy

$$c_1^2[M] = 8(1-g), \qquad c_2[M] = 4(1-g).$$

Hence, the Hirzebruch signature $\tau[M] = \frac{1}{3}\left(c_1^2[M] - 2c_2[M]\right)$ vanishes identically, and thus

$$\tau[M] = \frac{1}{12\pi^2}\int_M \left\{ \mathrm{tr}[(W^+)^2] - \mathrm{tr}[(W^-)^2] \right\} v = -\frac{1}{12\pi^2}\int_M \frac{Sc^2}{24} v,$$

which shows that $Sc = 0$, and thus M is flat.

Next, assume W^+ to be of Type Ib. Then $\ker(W^+ + \frac{4}{3}(\alpha - \gamma)\mathrm{Id})$ is one-dimensional and timelike, since otherwise the self-adjoint operator W^+ would diagonalize. The same occurs in case of Type II. In fact, if $\alpha \neq \beta$, then the eigenspace corresponding to $\frac{4}{3}(\alpha - \beta)$ defines an almost paraHermitian structure on M. Next, assume $\alpha = \beta$. Then it follows from [3] that $\alpha = \beta = 0$ and W^+ is two-step nilpotent. Hence $\mathrm{Im}W^+$ is one-dimensional and has an induced degenerate inner product. Therefore the restriction of the metric to $\mathrm{Im}W^+$ defines a one-dimensional null subspace. Recall from Remark 6 that unit sections $\Omega_\pm^+$ of $\Lambda_\pm^2$ of positive norm (equivalently, almost complex structures inducing opposite orientations) exist on M. Hence, for any null section $\Omega_\pm^0$ of $\Lambda_\pm^2$, put $\Omega_\pm^- = \frac{2}{\langle \Omega_\pm^0, \Omega_\pm^+ \rangle}\Omega_\pm^0 - \Omega_\pm^+$ to define a timelike section of $\Lambda_\pm^2$, and thus an almost paraHermitian structure with fundamental form $\Omega_\pm^-$. Finally, if W^+ is of Type III, then $(W^+)^2$ is two-step nilpotent. Hence existence of an almost paraHermitian structure follows then for all cases Ib,

II and III. Now, since we are assuming $W^- = 0$, (3) shows that

$$c_1^2[M] = -\frac{1}{2\pi^2}\int_M \left\{ \mathrm{tr}[(W^-)^2] + \frac{Sc^2}{48} \right\} v = -\frac{1}{2\pi^2}\int_M \frac{Sc^2}{48} v$$

and hence $Sc = 0$, which proves that (M, g) is Ricci flat. Finally, since no compact Ricci flat Type III Jordan-Osserman manifold may exist [13], either the sectional curvature is constant or otherwise the Jacobi operators are two-step nilpotent. □

3.2. **Proof of Theorem 2.** First of all, note that no four-dimensional Osserman metric exists whose Jacobi operators have three-distinct real eigenvalues, and moreover, the same holds true as concerns Osserman metrics with complex eigenvalues for the Jacobi operators [3]. Thus we must show there are no compact Osserman metrics with two-distinct real eigenvalues of signature $(--++)$. Let α and β denote the constant eigenvalues of the Jacobi operators, the later assumed to be of multiplicity two, and put $E_\alpha(X) = \langle X\rangle \oplus \ker(\mathcal{J}(X) - \alpha\langle X, X\rangle \mathrm{Id})$, the two dimensional subspace spanned by X and the eigenvector corresponding to the distinguished eigenvalue α.

Observe that, for any non-null vector X, the restriction of the metric tensor to $E_\alpha(X)$ must be non-degenerate, and thus either definite (of signature $(++)$ or $(--)$) or indefinite (of Lorentzian signature $(-+)$). Further the signature type of the E_α's cannot change from definite to indefinite, since otherwise it should pass through a degenerate case. Next, we will consider the two different possibilities.

If the induced metric on the E_α's is definite, then $\ker(\mathcal{J}(X) - \beta\langle X, X\rangle \mathrm{Id})$ is also definite, and thus the Jacobi operators are diagonalizable. This shows that M is locally an indefinite complex space form [3], and thus no such metrics exist in the compact case as shown in the proof of Theorem 1. Next, assume the induced metric on the E_α's is indefinite. Then, $\ker(\mathcal{J}(X) - \alpha\langle X, X\rangle \mathrm{Id})$ defines an almost paracomplex structure P on M, which makes (M, g, P) an almost paraHermitian manifold, and thus $c_1^2[M] = 0$, which proves Ricci flatness since $Sc = 0$, just proceeding as in the proof of Theorem 1. Finally we show that this cannot occur, thus finishing the proof. Indeed, observe that the eigenvalues α and β are in a ratio $1 : \frac{1}{4}$, as it was proved in [3] for the diagonalizable case, and moreover, it also follows as a consequence of the local description of the nondiagonalizable case given in [15]. Hence, $\alpha = 4\beta = 0$, which is a contradiction.

As a consequence of the above, a compact and simply connected Osserman 4-manifold M with metric of signature $(--++)$ has $\tau[M] = 0$ and $\chi[M] \le 0$, with equality if and only if the Jacobi operators are nilpotent. □

4. Compact Kähler-Osserman manifolds

If a compact complex surface admits an indefinite Kähler-Einstein metric, then it is one of the following [27]: i) a complex torus; ii) a Hyperelliptic surface; iii) a primary Kodaira surface; iv) a minimal ruled surface over a curve of genus $\mathfrak{g} \ge 2$, or v) a minimal surface of class VII_0. Moreover, the existence of the later metrics (case v)) is still an open problem.

However, a complete description of compact surfaces admitting an indefinite Kähler-Osserman metric follows now from [23, 27] and the previous results. Indeed, since $\tau[M] = b^+ - b^- = 0$, it follows that $\chi[M] = 2(1 - b_1 + b^-)$, and thus (since $\chi[M] \le 0$) $b_1 \ge 1 + b^-$ and $b_1 = 1 + b^-$ if and only if $\chi[M] = 0$. Now, an indefinite Kähler surface satisfies $H^2(M; \mathbb{R}) \ne 0$, from where $b^- = b^+ \ge 1$, and hence $b_1 \ge 2$.

This shows that no minimal surface of class VII is Kähler Osserman. Hence, the list of possible Kähler-Osserman metrics reduces to i) and iii) (see the proof of Theorem 1).

Recall that a *hypersymplectic* structure on a $4n$-dimensional manifold (M, g) of neutral signature is given by a triple (J_1, J_2, J_3) of skew-adjoint endomorphisms of the tangent bundle satisfying

$$J_1^2 = -\mathrm{Id}, \qquad J_2^2 = J_3^2 = \mathrm{Id}, \qquad J_1 J_2 = -J_2 J_1 = J_3$$

such that the corresponding 2-forms $\Omega_i(X, Y) = g(J_i X, Y)$ are closed. Equivalently, the tensor fields J_i are all parallel, and hence any hypersymplectic manifold is a Ricci-flat indefinite Kähler manifold. Therefore, any hypersymplectic structure is self-dual and Ricci flat, and thus Osserman. It was shown by Kamada [23] the existence of hypersymplectic structures on complex torus and primary Kodaira surfaces, and hence Fact 3 shows

Theorem 8. *A compact complex surface admits an indefinite Kähler-Osserman metric if and only if it is a complex torus or a primary Kodaira surface.*

Remark 9. The special significance of hypersymplectic structures in dimension four is given by the fact that any such structure is just a basis $\{\Omega_1, \Omega_2, \Omega_3\}$ of closed anti-self-dual 2-forms satisfying $-\Omega_1^2 = \Omega_2^2 = \Omega_3^2$ and $\Omega_i \wedge \Omega_j = 0$ for all $i \neq j$. Now, the endomorphisms $J_{12} = J_1 + J_2$ and $J_{13} = J_1 + J_3$ are skew-adjoint, nilpotent and moreover J_{12} and J_{13} are parallel. Therefore $\mathrm{Im} J_{12}$ and $\mathrm{Im} J_{13}$ are parallel degenerate distributions on M, which shows that (M, g) is a Walker manifold. Four-dimensional self-dual Ricci flat Walker metrics are known to be Riemannian extensions of torsion-free connections with skew-symmetric Ricci tensor [6]. Now,it follows from the analysis in [14] (see also [8]) that any hypersymplectic four-manifold is locally isometric to the cotangent bundle $T^*\Sigma$ of an affine surface Σ with torsion-free connection D given by ${\Gamma_{11}}^1 = -\partial_1\varphi$ and ${\Gamma_{22}}^2 = \partial_2\varphi$, for an arbitrary function φ. The metric g, in induced coordinates $(x^i, x_{i'})$ now reads as

$$g_\nabla = 2\, dx^i \circ dx_{i'} + (\phi_{ij} - 2x_{k'}{\Gamma_{ij}}^k) dx^i \circ dx^j$$

for some arbitrary symmetric $(0, 2)$-tensor field ϕ on Σ.

Further, it has been shown by Kamada [23] that any compact hypersymplectic surface admits two parallel null and orthogonal vector fields (i.e., the Walker metric is strict). A straightforward calculation now shows that any strictly Walker four-manifold is Einstein self-dual (hence a Riemannian extension as above) and moreover, the anti-self-dual Weyl curvature operator is two-step nilpotent. Now, a Riemannian extension has two-step nilpotent self-dual Weyl curvature operator if and only if the connection D is equiaffine [6], thus showing that *compact hypersymplectic manifolds are locally Riemannian extensions over flat affine surfaces.*

As an application of the above we have the following examples of compact Osserman manifolds generalizing those in [27]

Example 10. There are metrics on the 4-torus which are Walker signature $(2, 2)$ metrics which are Osserman and which are constructed as follows. Manifolds of this type were studied in this context first in [20]. Let $\{\theta_1, ..., \theta_4\}$ be the usual periodic parameters on the 4-torus. Let $g_{ij} = g_{ij}(\theta_3, \theta_4)$ be periodic functions for $3 \leq i, j \leq 4$. We consider the signature $(2, 2)$ Walker manifold $\mathcal{T} := (\mathbb{T}^4, g)$ where

g is given relative to the canonical frame ∂_{θ_i} by

$$g = 2d\theta_1 \circ d\theta_3 + 2d\theta_2 \circ d\theta_4 + g_{33} d\theta_3 \circ d\theta_3 + 2g_{34} d\theta_3 \circ d\theta_4 + g_{44} d\theta_4 \circ d\theta_4 . \tag{4}$$

As the only non-zero curvature is $R_{3443} = (2g_{34/34} - g_{33/44} - g_{44/33})/2$,

$$\mathcal{J}(\xi) : \mathrm{Span}\{\partial_3, \partial_4\} \to \mathrm{Span}\{\partial_1, \partial_2\} \to 0$$

and $\mathcal{T}$ is nilpotent Osserman. We assume R_{3443} does not vanish identically so $\mathcal{T}$ is not flat. The universal cover is a generalized plane wave manifold [21] and hence geodesically complete; thus $\mathcal{T}$ is itself geodesically complete. Clearly $\mathcal{M}$ is globally spacelike Jordan Osserman (or globally timelike Jordan Osserman) if and only if R_{3443} never vanishes. Since R_{3443} is in divergence form,

$$\int\int_{\mathbb{T}^2} R_{3443}(\theta_3, \theta_4) d\theta_3 d\theta_4 = 0 \,.$$

Thus R_{3443} must change sign and $\mathcal{T}$ is not globally Jordan Osserman. Finally observe that the metric (4) is locally isometric to a hypersymplectic metric as discussed in Remark 9.

References

[1] D. Alekseevsky, N. Blažić, N. Bokan, Z. Rakić; Self-duality and pointwise Osserman manifolds, *Arch. Math. (Brno)* **35** (1999), 193–201.

[2] S. Alexandrov, B. Pioline, S. Vandoren; Self-dual Einstein Spaces, Heavenly Metrics and Twistors, arXiv:0912.3406v1.

[3] N. Blažić, N. Bokan, Z. Rakić; Osserman pseudo-Riemannian manifolds of signature $(2, 2)$, *J. Aust. Math. Soc.* **71** (2001), 367–395.

[4] A. Bonome, R. Castro, E. García-Río, L. Hervella, Y. Matsushita; Pseudo-Chern classes and opposite Chern classes of indefinite almost Hermitian manifolds, *Acta Math. Hungar.* **75** (1997), 299–316.

[5] A. Borowiec, M. Francaviglia, I. Volovich; Anti-Kählerian manifolds, *Differential Geom. Appl.* **12** (2000), 281–289.

[6] M. Brozos-Vázquez, E. García-Río, P. Gilkey, S. Nikčević, R. Vázquez-Lorenzo; *The geometry of Walker manifolds*, Synthesis Lectures on Mathematics and Statistics **5**, Morgan & Claypool Publ., 2009.

[7] E. Calabi, L. Markus; Relativistic space forms, *Ann. Math.* **75** (1962), 63–76.

[8] E. Calviño-Louzao, E. García-Río, P. Gilkey, R. Vázquez-Lorenzo; The geometry of modified Riemannian extensions, *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* **465** (2009), 2023–2040.

[9] A. Chudecki, M. Przanowski; From hyperheavenly spaces to Walker and Osserman spaces I, *Classical Quantum Gravity* **25** (2008), no. 14, 145010, 18 pp.

[10] A. Chudecki, M. Przanowski; From hyperheavenly spaces to Walker and Osserman spaces II, *Classical Quantum Gravity* **25** (2008), no. 23, 235019, 22 pp.

[11] J. Davidov, G. Grantcharov, O. Mushkarov, M. Yotov; Para-hyperhermitian surfaces, Bull. Math. Soc. Sci. Math. Roumanie (N.S.) **52** (2009), 281–289.

[12] A. Derdzinski; Non-Walker self-dual neutral Einstein four-manifolds of Petrov type III, *J. Geom. Anal.* **19** (2009), 301-357.

[13] A. Derdzinski; Noncompactness and maximum mobility of type III Ricci-flat self-dual neutral Walker four-manifolds, *Quart. J. Math. Oxford*, doi:10.1093/qmath/hap033.

[14] A. Derdzinski; Connections with skew-symmetric Ricci tensor on surfaces, *Results Math.* **52** (2008), 223–245.

[15] J. C. Díaz-Ramos, E. García-Río, R. Vázquez-Lorenzo; Four-dimensional Osserman metrics with nondiagonalizable Jacobi operators, *J. Geom. Anal.* **16** (2006), 39–52.

[16] M. Dunajski, S. West; Anti-self-dual conformal structures in neutral signature, *Recent developments in pseudo-Riemannian geometry*, 113–148, ESI Lect. Math. Phys., Eur. Math. Soc., Zürich, 2008.

[17] E. García-Río, P. Gilkey, M. E. Vázquez-Abal, R. Vázquez-Lorenzo; Four-dimensional Osserman metrics of neutral signature, *Pacific J. Math.* **244** (2010), 21–36.
[18] E. García-Río, D. N. Kupeli, R. Vázquez-Lorenzo; *Osserman manifolds in semi-Riemannian geometry*, Lect. Notes Math. **1777**, Springer-Verlag, Berlin, Heidelberg, New York, 2002.
[19] P. Gilkey; *Geometric Properties of Natural Operators Defined by the Riemannian Curvature Tensor*, World Scientific Publishing Co., Inc., River Edge, NJ, 2001.
[20] P. Gilkey, R. Ivanova, T. Zhang; Szabo Osserman IP Pseudo-Riemannian manifolds, *Publ. Math. Debrecen* **62** (2003), 387–401.
[21] P. Gilkey, S. Nikčević; Generalized plane wave manifolds, *Kragujevac J. Math.* **28** (2005), 113–138.
[22] S. Ivanov, S. Zamkovoy; Parahermitian and paraquaternionic manifolds, *Differential Geom. Appl.* **23** (2005), 205–234.
[23] H. Kamada; *Self-dual Kähler metrics of neutral signature on complex surfaces*, Tohoku University, Sendai, 2002. *Tohoku Mathematical Publications*, **24**. Tohoku University, Mathematical Institute, Sendai, 2002. ii+94 pp.
[24] P. R. Law; Neutral Einstein metrics in four dimensions, *J. Math. Phys.* **32** (1991), 3039–3042.
[25] P. R. Law, Y. Matsushita; Hitchin-Thorpe-type inequalities for pseudo-Riemannian 4-manifolds of metric signature $(++--)$, *Geom. Dedicata* **87** (2001), 65–89.
[26] Y. Matsushita; Fields of 2-planes and two kinds of almost complex structures on compact 4-dimensional manifolds, *Math. Z.* **207** (1991), 281–291.
[27] J. Petean; Indefinite Kähler-Einstein metrics on compact complex surfaces, *Comm. Math. Phys.* **189** (1997), 227–235.

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